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ON A CLASS OF SECOND ORDER VARIATIONAL PROBLEMS WITH CONSTRAINTS

BY

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ABSTRACT

We study the structure of optimal solutions for a class of constrained, second order variational problems on bounded intervals. We show that, for intervals of length greater than some positive constant, the optimal solutions are bounded in C^1 by a bound independent of the length of the interval. Furthermore, for sufficiently large intervals, the 'mass' and 'energy' of optimal solutions are almost uniformly distributed.

Introduction

In this paper we investigate the structure of optimal solutions of variational problems associated with a class of functionals of the form

$$I^f(D;w) = \int_D f(w(t), w'(t), w''(t))dt, \quad \forall w \in W^{2,1}(D), \tag{0.1}$$

where D is a bounded interval on the real line and $f \in C(R^3)$ belongs to a space $\mathfrak{M}$, to be described below, such that $I^f(D;\cdot)$ is an extended real functional on $W^{2,1}(D)$ which may obtain the value $+\infty$ but is bounded from below. Specifically we shall study the problems

$$(P_D^f) \qquad \inf\{J^f(D;w) \colon w \in W^{2,1}(D)\}$$

and

$$(P_D^f)_a \qquad \inf\{J^f(D;w) \colon w \in W^{2,1}(D), \langle w\rangle_D = a\},$$

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where

$$J^f(D;w) = \frac{1}{|D|} I^f(D;w), \quad \langle w \rangle_D = \frac{1}{|D|} \int_D w\,dx. \tag{0.2}$$

We shall also consider the corresponding problems with prescribed boundary values for $X_w := (w, w')$,

$(P_D^f)^{x,y}$
$$\inf\{J^f(D;w) \colon w \in W^{2,1}(D),\ X_w(T_1) = x,\ X_w(T_2) = y\}, \quad D = (T_1, T_2)$$

and

$(P_D^f)_a^{x,y}$ $\inf\{J^f(D;w) : w \in W^{2,1}(D),\ X_w(T_1) = x,\ X_w(T_2) = y,\ \langle w \rangle_D = a\}.$

It will be assumed that f satisfies conditions which guarantee that each of the above problems possesses a minimizer for every bounded interval D, and we shall be interested in the structure of these minimizers for large $|D|$. As in [8, 9, 10] this study is based on the relation between problems (P_D^f) and $(P_D^f)_a$ on large intervals and the corresponding limiting problems on $\mathbb{R}$:

(P^f) $\inf\{J^f(w) \colon w \in W^{2,1}_{loc}(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})\},$

$(P^f)_a$ $\inf\{J^f(w) \colon w \in W^{2,1}_{loc}(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R}),\ \langle w \rangle = a\},$

where

$$J^f(w) = \liminf_{T \to \infty} J^f((-T,T);w), \qquad \langle w \rangle = \lim_{T \to \infty} \langle w \rangle_{(-T,T)}. \tag{0.3}$$

Similarly we define

$$J_+^f(w) = \liminf_{T \to \infty} J^f((0,T);w), \qquad \langle w \rangle_+ = \lim_{T \to \infty} \langle w \rangle_{(0,T)}, \tag{0.4}$$

and the one-sided limiting problems (P_+^f), $(P_+^f)_a$.

Note that problem $(P_D^f)_a$ on an interval $D = (0,T)$ is equivalent to problem

$(P^f)_a^\epsilon$ $$\inf\left\{\int_0^1 f(v, \epsilon v', \epsilon^2 v'')\,ds \colon v \in W^{2,1}(0,1), \int_0^1 v = a\right\},$$

where $\epsilon = 1/T$ and $v(s) = w(sT)$, $w \in W^{2,1}(0,T)$. Therefore our study is tantamount to the study of a class of **singular perturbation problems**.

The constrained variational problems on bounded intervals were conceived as models for determining the thermodynamical equilibrium states of unidimensional bodies involving 'second order' materials (see [4]). In the special case

$$f(u,u',u'') = u''^2 - bu'^2 + \psi(u), \tag{0.5}$$

where ψ is a potential such as $\psi(u) = (u^2-1)^2$, problem $\left(P^f\right)_a^\epsilon$ becomes

$$\inf\left\{ \tfrac{1}{2}\int_{-1}^{1} (\epsilon^2 v''^2 - b\epsilon v'^2 + \psi(v))\,ds \colon v \in W^{2,1}(-1,1),\ \tfrac{1}{2}\int_{-1}^{1} v = a \right\}.$$

We note that for $\epsilon = 0$ this reduces to the Gibbs free energy model, while for $\epsilon > 0$ and $b = 0$, (0.5) reduces to a second order version of the van der Waals model.

The unconstrained limiting problem $\left(P^f\right)$ was studied in [7], [14, 15, 16], [10] and [11]. The constrained limiting problem $\left(P^f\right)_a$ was studied in [4], [8] and [9]. The relation between the problems on bounded large intervals and the corresponding limiting problems was first studied in [8] and [9] (for constrained problems) and in [10] (for unconstrained problems). Equations related to example (0.5) were studied by many authors. For example, the stationary states of the extended Fisher-Kolmogorov equation, namely $-\gamma u^{(iv)} + u'' + (u - u^3) = 0$, $\gamma > 0$, were studied in [12, 13]. For other related studies see [3, 5] and the references mentioned therein.

We turn now to the definition of the spaces of integrands considered in the present paper. Put

$$\begin{aligned}\mathfrak{A} = \{ & f \in C(R^3) \colon |f(x_1,x_2,x_3)| \to \infty \text{ as } |x_3| \to \infty, \\ & \text{uniformly with respect to } (x_1,x_2) \text{ in compact sets}\}.\end{aligned} \tag{0.6}$$

$\mathfrak{A}$ will be equipped with the uniformity determined by the base

$$\begin{aligned} E(N,\epsilon) = \{ (f,g) \in \mathfrak{A}\times\mathfrak{A} \colon & \text{(i)} |f(x) - g(x)| \le \epsilon, \text{ if } |x_i| \le N,\ i = 1,2,3, \\ & \text{(ii)} 1 - \epsilon \le \frac{|f(x)|+1}{|g(x)|+1} \le 1 + \epsilon, \text{ if } |x_1|, |x_2| \le N \\ & \forall x = (x_1,x_2,x_3) \in R^3\}, \end{aligned} \tag{0.7}$$

where N and ϵ are positive numbers. It is easy to verify that the uniform space $\mathfrak{A}$ is metrizable and complete.

The space of integrands $\mathfrak{M}$ considered in this paper is a subspace of $\mathfrak{A}$ which depends on several parameters, namely $\bar{a} = (a_1, \dots, a_4) \in \mathbb{R}^4$, $a_i > 0$ and $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$1 \le \beta < \alpha, \quad \beta \le \gamma, \quad 1 < \gamma. \tag{0.8}$$

A function $f \in \mathfrak{A}$ belongs to $\mathfrak{M} = \mathfrak{M}(\alpha, \beta, \gamma, \bar{a})$ iff it satisfies the following properties:

$$\begin{aligned}
&\text{(i) } f \in C^2(R^3),\ \partial f/\partial x_2 \in C^2(R^3),\ \partial f/\partial x_3 \in C^3(R^3),\\
&\text{(ii) } \partial^2 f/\partial x_3^2 > 0,\\
&\text{(iii) } f(x) \ge a_1|x_1|^\alpha - a_2|x_2|^\beta + a_3|x_3|^\gamma - a_4,\\
&\text{(iv) } (|f| + |\nabla f|)(x) \le M_f(|x_1| + |x_2|)(1 + |x_3|^\gamma), \quad \forall x \in R^3,
\end{aligned} \tag{0.9}$$

where $M_f : [0, \infty) \mapsto [0, \infty)$ is a continuous function depending on f. The closure of $\mathfrak{M}$ in $\mathfrak{A}$ will be denoted by $\bar{\mathfrak{M}}$.

Assumption (0.9) implies that if $f \in \mathfrak{M}$ and D is a bounded interval, then $I^f(D; \cdot)$ is an extended real valued functional on $W^{2,1}(D)$ with range $(-\infty, +\infty]$. For $v \in W^{2,1}(D)$, $I^f(D; v) < +\infty$ if and only if $v \in W^{2,\gamma}(D)$. In addition, (0.8) and (0.9) imply that this functional is bounded below (see (0.11)).

Given a function $f \in \bar{\mathfrak{M}}$ and a bounded interval D with $|D| = T$ put

$$\begin{aligned}
\mu^f &= \inf(P^f), & \varphi^f(a) &= \inf(P^f)_a,\\
\mu^f_T &= \inf(P^f_D), & \varphi^f_T(a) &= \inf(P^f_D)_a,\\
\hat{\mu}^f_T(x, y) &= \inf(P^f_D)^{x,y}, & \hat{\varphi}^f_T(a; x, y) &= \inf(P^f_D)^{x,y}_a,
\end{aligned} \tag{0.10}$$

for every $x, y \in \mathbb{R}^2$ and $a \in \mathbb{R}$. The functions φ^f, φ^f_T, $\hat{\varphi}^f_T$, $\hat{\mu}^f_T$ are called **response functions** of the respective problems. Some properties of φ^f are described in Proposition 0.1 below. Each of the other functions is continuous in all its variables (e.g., $\hat{\mu}^f_T \in C(\mathbb{R}^4)$) and tends to infinity as its argument tends to infinity in Euclidean norm (see [8, Lemma 1.3]).

Conditions (0.8) and (0.9)(iii) imply that there exists a constant b_0 such that, for every $f \in \bar{\mathfrak{M}}$ and every $T \ge 1$,

$$I^f(0, T, v) \ge \int_0^T \left(\frac{1}{2} a_3 |v''|^\gamma + a_1 |v|^\alpha\right) dt - b_0 T, \quad \forall v \in W^{2,\gamma}(0, T) \tag{0.11}$$

(see [10, Lemma 2.2]). Obviously b_0 depends on the parameters defining $\mathfrak{M}$. This coercivity property and conditions (0.8) and (0.9) imply, by standard existence theory, that each of the problems $\left(P^f_{(0,T)}\right)$, $\left(P^f_{(0,T)}\right)_a$, $\left(P^f_{(0,T)}\right)^{x,y}$, $\left(P^f_{(0,T)}\right)^{x,y}_a$ possesses a minimizer for every $T>0$, every $x,y\in\mathbb{R}^2$ and every $a\in\mathbb{R}$. The existence of minimizers for the unconstrained problem $\left(P^f\right)$ was established in [7]. Moreover, it was shown that if $\mu^f<\inf_{\mathbb{R}^2} f(w,0,s)$ then $\left(P^f\right)$ possesses a *periodic* minimizer. The restriction on f was later removed in [14]. The existence of minimizers for the constrained problems $\left(P^f\right)_a$ was established in [4]. Some additional results of [4] are summed up in the next proposition.

PROPOSITION 0.1: *Assume that $f\in\mathfrak{M}$. Then:*
(i) *The function $a\mapsto\varphi^f(a)$ is finite everywhere and convex on $\mathbb{R}$.*
(ii) *If a is an exposed point with respect to φ^f, i.e.*

$$(0.12)\qquad \exists\lambda\in\mathbb{R}:\quad \varphi^f(s)>\varphi^f(a)+\lambda(s-a),\quad \forall s\in\mathbb{R}\setminus\{a\},$$

then $(P^f)_a$ possesses a periodic minimizer.
(iii) $\varphi(a)/|a|\to\infty$ *as* $|a|\to\infty$.

A study of the properties of minimimizers of the constrained problem $\left(P^f_D\right)_a$ on bounded intervals was initiated in [8], [9]. These papers were concerned with the relation between the problems on large bounded intervals and the formally limiting problem $(P^f)_a$. It was shown that this relation can be employed in order to derive properties of minimizers on large intervals on the basis of information concerning the limiting problem. These papers dealt with integrands of the form (0.5), where ψ is a potential satisfying certain growth and monotonicity conditions at $\pm\infty$ including the coercivity condition $\psi(u)/u^2\to\infty$ as $|u|\to\infty$. The standard example is provided by the double well potential, $\psi(u)=(u^2-1)^2$.

In the present paper we continue this study extending it to the general class of integrands $\mathfrak{M}$. In addition we shall be concerned not only with minimizers of $(P^f_D)_a$ but also with the wider family of *almost minimizers*, which will be defined below. The study of our variational problems in this wider context requires new techniques.

For the statement of our main results we need the following definition.

Definition 0.2: Let $f\in\mathfrak{M}$, let D be a bounded interval and let $\mathcal{P}^f_D$ stand for any of the variational problems involving the functional $J^f(D,\cdot)$ defined above. Given $\delta>0$ and $u\in W^{2,\gamma}(D)$ we shall say that u is a δ-**approximate minimizer** of problem $\mathcal{P}^f_D$ if
(i) u satsifies the constraints and boundary conditions associated with the problem, and
(ii) $J^f(D;u)\le\inf\mathcal{P}^f_D+\delta/|D|$.

At this point it is convenient to introduce also the following terminology and notation.

Definition 0.3: Let $f \in \mathfrak{M}$ and $\xi \in \mathbb{R}$.
(a) The subdifferential of the response function φ^f at ξ will be denoted by $\partial\varphi^f(\xi)$. The set

$$\partial'\varphi^f(\xi) = \{\lambda \in \mathbb{R} : \quad \varphi^f(s) > \varphi^f(\xi) + \lambda(s-a), \quad \forall s \in \mathbb{R} \setminus \{\xi\}\} \tag{0.13}$$

will be called the **reduced subdifferential** of φ^f at ξ.
(b) ξ is an **exposed point** relative to φ^f if $\partial'\varphi^f(\xi) \neq \emptyset$.
(c) ξ is an **extremal point** relative to φ^f if $(\xi, \varphi^f(\xi))$ is extremal relative to the epigraph of φ^f.
(d) Given $\xi \in \mathbb{R}$ and $\lambda \in \partial\varphi^f(\xi)$, put

$$\mathcal{E}_f(\lambda) = \{\eta \colon \varphi^f(\eta) - \varphi^f(\xi) = \lambda(\eta - \xi)\}.$$

If φ^f is differentiable at ξ and $\lambda = (\varphi^f)'(\xi)$, put $\mathcal{E}_f^*(\xi) := \mathcal{E}_f(\lambda)$.

Clearly $\mathcal{E}_f(\lambda)$ is a closed interval containing ξ and Proposition 0.1 implies that this interval is bounded. If $\lambda \in \partial'\varphi^f(\xi)$, then the interval reduces to one point. Note that, by Proposition 0.1, for every real λ there exists ξ such that $\lambda \in \partial\varphi^f(\xi)$.

Our first main result concerns the uniform boundedness of approximate minimizers.

THEOREM I: *Let $f \in \mathfrak{M}$. For every $\lambda \in \mathbb{R}$ and every two positive numbers δ, τ there exists a number $C = C_f(\lambda;\, \delta, \tau)$ such that the following statement holds.*

Let $v \in W^{2,\gamma}(0,T)$, $\xi = \langle v \rangle_{(0,T)}$ and $\lambda \in \partial\varphi^f(\xi)$. If $T \geq \tau$ and v satisfies

$$J^f(0,T;v) \leq \varphi_T^f(\xi) + \delta/T, \tag{0.14}$$

i.e. v is a δ-approximate minimizer of $\left(P^f_{(0,T)}\right)_\xi$, then

$$\|v\|_{C^1[0,T]} \leq C_f(\lambda;\, \delta, \tau) \quad \text{and} \quad \sup_{0 \leq s \leq T-1} I^f((s,s+1);v) \leq C_f(\lambda;\, \delta, \tau). \tag{0.15}$$

Remark: For integrands f of the form (0.5), this result was established in [8] with respect to *minimizers* of $\left(P^f_{(0,T)}\right)_\xi$. In that case it was shown that, for every compact set K, the bound C can be chosen independently of $\xi = \langle v \rangle_{(0,T)}$ for $\xi \in K$.

Our second result concerns the uniform distribution of mass and energy of approximate minimizers in sufficiently large intervals.

THEOREM II: *Suppose that $f \in \mathfrak{M}$ and $\lambda \in \mathbb{R}$. Then, given two positive numbers ϵ, δ, there exists $L = L_f(\lambda; \epsilon, \delta) > 0$ such that the following statement holds.*

Let $v \in W^{2,\gamma}(0,T)$, $\xi = \langle v \rangle_{(0,T)}$ and $\lambda \in \partial\varphi^f(\xi)$. If v satisfies (0.14) and $T \geq L$, then

$$\text{(0.16)} \qquad \operatorname{dist}(\langle v \rangle_D, \mathcal{E}_f(\lambda)) \leq \epsilon, \quad \operatorname{dist}(J^f(D; v), \varphi^f(\mathcal{E}_f(\lambda)) \leq \epsilon,$$

for every interval $D \subset (0,T)$ such that $|D| \geq L$. In particular, if ξ is an exposed point and v is a δ-approximate minimizer of $\left(P^f_{(0,T)}\right)_\xi$, then

$$\text{(0.17)} \qquad |\langle v \rangle_D - \xi| \leq \epsilon, \qquad |J^f(D; v) - \varphi^f(\xi)| \leq \epsilon.$$

Remark: For integrands f of the form (0.5), this result was established in [9] with respect to *minimizers* of $\left(P^f_{(0,T)}\right)_\xi$. In that case it was shown that, for every compact interval K such that $\inf_K(f(\cdot, 0, 0) - \varphi^f) > 0$, L can be chosen independently of $\xi = \langle v \rangle_{(0,T)}$ for $\xi \in K$.

Our third result provides a more precise estimate for a related energy functional.

THEOREM III: *Assume $f \in \mathfrak{M}$ and $\lambda \in \mathbb{R}$. Then, given $\tau > 0$ and $\delta \geq 0$, there exists $C = C_f(\lambda; \tau, \delta) > 0$ such that the following statement holds.*

Let $v \in W^{2,\gamma}(0,T)$, $\xi = \langle v \rangle_{(0,T)}$ and $\lambda \in \partial\varphi^f(\xi)$. If v satisfies (0.14) and $T \geq \tau$, then

$$\text{(0.18)} \qquad |(J^f(D; v) - \lambda\langle v \rangle_D) - (\varphi^f(\xi) - \lambda\xi)| \leq C/|D|,$$

for every interval $D \subset (0,T)$. In particular, for every $\xi \in \mathbb{R}$ and $\lambda \in \partial\varphi^f(\xi)$,

$$\text{(0.19)} \qquad |\varphi^f_T(\xi) - \varphi^f(\xi)| \leq C_f(\lambda; \tau, 0)/T.$$

Remark: For integrands f of the form (0.5), inequality (0.19) was established in [9]. In that case it was shown that, for every compact interval K such that $\inf_K(f(\cdot, 0, 0) - \varphi^f)$ is positive, C_f can be chosen independently of ξ in K.

The plan of the paper is as follows. In section 1 we establish the continuity of the response functions $(\xi, T, f) \mapsto \varphi^f_T(\xi)$, $(x, y, T, f) \mapsto \varphi^f_T(x, y)$ and $(\xi, x, y, T, f) \mapsto \hat{\varphi}^f_T(\xi, x, y)$ and some related boundedness results. In addition we show that they are Lipschitz continuous with respect to ξ, x, y and Hölder continuous with respect to T, uniformly with respect to f in appropriate subsets of $\mathfrak{M}$. In section 2 we present some background results and establish the existence of periodic minimizers of problem $(P^f)_\xi$ whenever ξ is an *extremal* point of φ^f. Section 3 is devoted to the proof of Theorem I. In section 4 we prove Theorems II and III. In addition we show that Theorems I–III extend to problems $\left(P^f_{(0,T)}\right)^{x,y}$ and $\left(P^f_{(0,T)}\right)^{x,y}_\xi$.

1. Continuity of response functions in bounded intervals

In this section we establish Lipschitz continuity of response functions and uniform boundedness results for families of minimizers in bounded intervals.

We start with some notation. Given $f \in \mathfrak{M}$, $T > 0$, $\xi \in \mathbb{R}$ and $x, y \in \mathbb{R}^2$ let $\mathcal{S}^f_T(\xi)$ (resp. $\mathcal{S}^f_T(x,y)$, $\mathcal{S}^f_T(\xi,x,y)$) denote the set of minimizers of problem $\left(P^f_{(0,T)}\right)_\xi$ (resp. $\left(P^f_{(0,T)}\right)^{x,y}$, $\left(P^f_{(0,T)}\right)^{x,y}_\xi$). Further, put

$$\text{(1.1)}\quad \Lambda^f_1(\xi,T) = \varphi^f_T(\xi),\ \ \Lambda^f_2(x,y,T) = \hat{\mu}^f_T(x,y),\ \ \Lambda^f_3(\xi,x,y,T) = \hat{\varphi}^f_T(\xi;x,y).$$

Finally let $E_1 = \mathbb{R}$, $E_2 = \mathbb{R}^2 \times \mathbb{R}^2$, $E_3 = \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2$ so that Λ_i is a function on $E_i \times (0,\infty)$.

The following is the main result of this section:

THEOREM 1.1: *Let Λ^f_i, $i = 1,2,3$ be as in (1.1). Let $D \subset (0,\infty)$ be a compact interval and let K_i be a compact subset of E_i, $i = 1,2,3$. Then:*

(i) *Suppose that $\mathcal{F}_0 \subset \mathfrak{M}$ is uniformly bounded in compact sets. Then the set $\mathcal{S}^f_T(z)$ is bounded in $W^{2,\gamma}(0,T)$ by a constant independent of $f \in \mathcal{F}_0$, $T \in D$, $z \in K_i$ $(i = 1,2,3)$.*

(ii) *For every $f \in \mathfrak{M}$, the function $(z,T,f) \mapsto \Lambda^f_i(z,T)$ is continuous on $E_i \times (0,\infty) \times \mathfrak{M}$. In addition the function is uniformly continuous on $K_i \times D \times \mathcal{F}$, for any family of functions $\mathcal{F} \subset \mathfrak{M}$ which satisfies condition (0.9)*(iv) *uniformly, i.e., $\exists M \in C[0,\infty)$ such that*

$$\text{(1.2)}\quad |f(x)| + |\nabla f(x)| \le M(|x_1| + |x_2|)(1 + |x_3|^\gamma), \quad \forall x \in R^3,\ \forall f \in \mathcal{F}.$$

(iii) *Let $\mathcal{F}$ be as in* (ii). *Then there exists a constant $M_i = M_i(D, K_i, \mathcal{F})$ such that, for every $f \in \mathcal{F}$,*

$$\text{(1.3)}\qquad \left|\Lambda^f_i(z,T) - \Lambda^f_i(z',T')\right| \le M_i(|z - z'| + |T - T'|^{1/\gamma'}),$$
$$\forall T, T' \in [\tau, \tau'],\ z, z' \in K_i,$$

where $1/\gamma + 1/\gamma' = 1$ with γ as in the definition of $\mathfrak{M}$ (see (0.9)).

The proof is based on several lemmas.

LEMMA 1.2: *Let D, K_i, $\mathcal{F}_0$ and $\mathcal{F}$ be as in the statement of the theorem. Then:* (i) *The set $\mathcal{S}_T^f(z)$ is bounded in $W^{2,\gamma}(0,T)$ by a constant independent of $f \in \mathcal{F}_0$, $T \in D$, $z \in K_i$ $(i = 1,2,3)$.* (ii) *There exists a constant $M_i' = M_i'(D, K_i, \mathcal{F})$ such that, for $i = 1,2,3$,*

$$(1.4) \qquad \left|\Lambda_i^f(z,T) - \Lambda_i^f(z',T)\right| \leq M_i'|z - z'|, \quad \forall T \in D,\ z, z' \in K_i,\ f \in \mathcal{F}.$$

In some special cases, the continuity of $\Lambda_i^f(\cdot, T)$ (but not the Lipschitz property) was proved in [7] and [8]. Another related result was obtained in [10]. Our proof, given in Appendix A, employs an idea of [8, Lemma 1.3].

LEMMA 1.3: *Under the same assumptions as before, there exists a constant M_i^* depending only on $D, K_i, \mathcal{F}$ such that, for $i = 1,2,3$,*

$$(1.5) \quad \left|\Lambda_i^f(z,T_1) - \Lambda_i^f(z,T_2)\right| \leq M_i^*|T_1 - T_2|^{1/\gamma'}, \quad \forall T_1, T_2 \in D,\ z \in K_i,\ f \in \mathcal{F}.$$

Proof: To fix ideas we shall prove (1.5) for $i = 3$. The proof is the same in the other two cases. In what follows $c_0, c_1, \ldots$ denote constants depending only on $D, K_3, \mathcal{F}$. In particular we denote by c_0 a bound of $\mathcal{S}_T^f(z)$ in $C^1[0,T]$ which, in view of Lemma 1.2, can be chosen to be independent of $z \in K_3$, $T \in D$, $f \in \mathcal{F}$.

Suppose that $T_1 < T_2$. Let $v \in \mathcal{S}_{T_1}^f(z)$, $z = (\xi, x, y) \in K_3$. Let $\bar{v}$ be the extension of v to $[0, T_2]$ which is linear in $[T_1, T_2]$ and satisfies $X_{\bar{v}}(T_1) = X_v(T_1) = y$. Put $\bar{y} = X_{\bar{v}}(T_2)$, $\bar{\xi} = \langle \bar{v} \rangle_{(0,T_2)}$ and $\bar{z} = (\bar{\xi}, x, \bar{y})$. Then $\bar{v} \in W^{2,\gamma}(0, T_2)$ and

$$(1.6) \qquad \Lambda_3^f(\bar{z}, T_2) \leq \frac{1}{T_2} I^f(0, T_2, \bar{v}) \leq \frac{T_1}{T_2}\Lambda_3^f(z, T_1) + \frac{1}{T_2} I^f(T_1, T_2, \bar{v}).$$

Since $\bar{v}$ restricted to $[T_1, T_2]$ depends only on y,

$$(1.7) \qquad \frac{1}{T_2}\left|I^f(T_1, T_2, \bar{v})\right| \leq c_1(T_2 - T_1), \quad |y - \bar{y}| + |\xi - \bar{\xi}| \leq c_1(T_2 - T_1),$$

where c_1 depends only on K_3, $\mathcal{F}$. By (1.6), (1.7) and Lemma 1.2(ii),

$$(1.8) \qquad \begin{aligned} \Lambda_3^f(z, T_2) &\leq \Lambda_3^f(\bar{z}, T_2) + M_3|z - \bar{z}| \\ &\leq \Lambda_3^f(z, T_1) + M_3|z - \bar{z}| + c_1(T_2 - T_1) \\ &\leq \Lambda_3^f(z, T_1) + c_2(T_2 - T_1). \end{aligned}$$

Let $w \in \mathcal{S}_{T_2}^f(z)$ and put $y' = X_w(T_1)$, $\xi' = \langle w \rangle_{(0,T_1)}$ and $z' = (\xi', x, y')$. In view of Lemma 1.2(i) there exists a constant c_2 (independent of T_1, T_2, w) such that

$$(1.9) \qquad |\xi' - \xi| + |w(T_1) - w(T_2)| \leq c_2(T_2 - T_1)$$

and

$$|w'(T_1) - w'(T_2)| \le \int_{T_1}^{T_2} |w''(s)|\,ds \le (T_2 - T_1)^{1/\gamma'} \Big(\int_{T_1}^{T_2} |w''(s)|^\gamma\,ds\Big)^{1/\gamma}$$
$$\le c_2(T_2 - T_1)^{1/\gamma'}. \tag{1.10}$$

Choose $N > 0$ sufficiently large so that

$$|b_1| \le c_0,\ |b_2| \le c_0,\ b_3 \ge N \Longrightarrow f(b_1, b_2, b_3) > 0, \quad \forall f \in \mathfrak{M}.$$

Let χ be a function on $(0, T_2)$ defined by:

$$\chi(t) = 0 \quad \text{if } w''(t) \ge N \quad \text{and} \quad \chi(t) = 1 \quad \text{otherwise.}$$

Put

$$A_1 = \int_{T_1}^{T_2} \chi(t) f(w, w', w'')\,ds, \quad A_2 = \int_{T_1}^{T_2} (1 - \chi(t)) f(w, w', w'')\,ds.$$

Then, $A_2 \ge 0$ and there exists a positive constant c_3 such that

$$A = \int_{T_1}^{T_2} f(w, w', w'')\,ds \ge A_1 \ge -c_3(T_2 - T_1). \tag{1.11}$$

By (1.9), (1.10) and Lemma 1.2(ii), there exists a constant c_4 such that

$$\Big|\Lambda_3^f(z, T_1) - \Lambda_3^f(z', T_1)\Big| \le M_3|z' - z| \le c_4(T_2 - T_1)^{1/\gamma'}. \tag{1.12}$$

By (1.11),

$$\Lambda_3^f(z', T_1) = \frac{1}{T_1} I^f(0, T_1, w) \le \frac{1}{T_1}\big(I^f(0, T_2, w) + c_3(T_2 - T_1)\big)$$
$$\le \Lambda_3^f(z, T_2) + c_3'(T_2 - T_1). \tag{1.13}$$

By (1.12) and (1.13), there exists a constant c_5 such that

$$\Lambda_3^f(z, T_1) \le \Lambda_3^f(z, T_2) + c_5(T_2 - T_1)^{1/\gamma'}. \tag{1.14}$$

Finally, (1.8) and (1.14) imply (1.5) for $i = 3$. ∎

Proof of Theorem 1.1: Statements (i) and (iii) follow immediately from Lemmas 1.2 and 1.3. Therefore it remains to prove (ii). The following fact is a consequence of [10, Lemma 2.7] and statement (i) above:

Let $g \in \mathfrak{M}$, $z \in E_i$, $T \in (0,\infty)$. For every $\epsilon > 0$ there exist positive numbers N, δ such that

$$\left|\Lambda_i^f(z,T) - \Lambda_i^g(z,T)\right| \leq \epsilon, \quad \forall f \in E_g(N,\delta) = \{f \in \mathfrak{M} : (f,g) \in E(N,\delta)\},$$

with $E(N,\delta)$ as in (0.7).

In addition, from the proof of [10, Lemma 2.7], it is easy to see that N, δ can be chosen independently of g, z, T for $g \in \mathcal{F}$, $z \in K_i$, $T \in D$. Consequently, for every $z \in E_i$, $T \in (0,\infty)$, the function $f \mapsto \Lambda_i^f(z,T)$ is continuous. Furthermore, this function is uniformly continuous at elements $g \in \mathcal{F}$. These facts and statement (iii) imply (ii). ∎

2. Auxiliary results

We start with some notation and background results. We assume throughout that f is a function in $\mathfrak{M}$.

(a) The set of minimizers of (P^f) (resp. $(P^f)_\xi$) will be denoted by $\mathcal{S}(f)$ (resp. $\mathcal{S}(\xi; f)$).

The set of periodic solutions of problem (P^f) (resp. $(P^f)_\xi$) will be denoted by $\tilde{\mathcal{S}}(f)$ (resp. $\tilde{\mathcal{S}}(\xi; f)$). If $u \in \tilde{\mathcal{S}}(f)$, its minimal period will be denoted by $\tau(u)$.

In general $\tilde{\mathcal{S}}(\xi; f)$ may be empty, but if ξ is an exposed point $\tilde{\mathcal{S}}(\xi; f) \neq \emptyset$.

(b) For $\lambda \in \mathbb{R}$, put $f_\lambda(u,u',u'') = f(u,u',u'') - \lambda u$. Then $J^{f_\lambda}(D;u) = J^f(D;u) - \lambda\langle u\rangle_D$.

(c) In view of the convexity of φ^f (see Proposition 0.1) it is easy to verify that

$$(2.1) \qquad \begin{array}{ll} \mu^{f_\lambda} = \varphi^f(\xi) - \lambda\xi, & \forall\lambda \in \partial\varphi^f(\xi),\ \forall\xi \in \mathbb{R}, \\ \mathcal{S}(\xi; f) \subseteq \mathcal{S}(f_\lambda), & \forall\lambda \in \partial\varphi^f(\xi),\ \forall\xi \in \mathbb{R}, \\ \mathcal{S}(\xi; f) = \mathcal{S}(f_\lambda), & \forall\lambda \in \partial'\varphi^f(\xi),\ \forall\xi \in \mathbb{R}. \end{array}$$

In particular $\mu^f = \inf \varphi^f$. In addition,

$$(2.2) \qquad \mathcal{S}(f_\lambda) = \bigcup\{\mathcal{S}(\xi; f) : \lambda \in \partial\varphi^f(\xi)\}, \quad \forall\lambda \in \mathbb{R}.$$

(d) Applying the method of mixtures of [4, sect. 2], it is not difficult to see that

$$(2.3) \qquad \begin{aligned} \mu^f &= \inf(P^f) = \inf(P_+^f), \\ \varphi^f(\xi) &= \inf(P^f)_\xi = \inf(P_+^f)_\xi. \end{aligned}$$

(e) Suppose that $u \in W_{loc}^{2,\gamma}(\mathbb{R}_+) \cap W^{1,\infty}(\mathbb{R}_+)$. We shall say that u is **c-optimal** relative to f if, for every bounded interval $D \subset \mathbb{R}_+$, u is a minimizer of $(P_D^f)^{x,y}$, where x, y are the values of X_u at the end points of D. By [10, Lemma 2.6] such a function is necessarily a minimizer of (P_+^f). However, if the boundedness assumption is dropped, this conclusion may not remain valid.

In [7] a c-optimal minimizer was called a 'minimal energy configuration'. The concept was previously used in [1], with respect to a discrete model.

(f) The following result was established in [7, sect. 4]. A discrete version was previously obtained in [6].

For every $f \in \mathfrak{M}$ there exists a function $\pi^f \in C(\mathbb{R}^2)$ such that

$$\hat{\mu}_T^f(x,y) \geq T\mu^f + \pi^f(x) - \pi^f(y), \quad \forall x, y \in \mathbb{R}^2,\ \forall T > 0. \tag{2.4}$$

Furthermore, for every $T > 0$ and every $x \in \mathbb{R}^2$ there exists $y \in \mathbb{R}^2$ such that equality holds.

(g) For $D = (T_1, T_2)$ and $v \in W^{2,\gamma}(D)$ put

$$\Gamma^f(D; v) = I^f(D; v) - |D|\mu^f + \pi^f(X_v(T_2)) - \pi^f(X_v(T_1)). \tag{2.5}$$

Γ^f will be called the **modified energy** functional. By (2.4) this functional is non-negative. If $v \in W_{loc}^{2,\gamma}(\mathbb{R}_+) \cap W^{1,\infty}(\mathbb{R}_+)$, we shall say that it is **f-perfect** if $\Gamma^f(D; v) = 0$ for every bounded interval $D \subset \mathbb{R}_+$. Obviously, every f-perfect function is a c-optimal minimizer of (P_+^f).

We turn now to the main result of this section.

THEOREM 2.1: *Let $f \in \mathfrak{M}$. If ξ is an extremal point of the response function φ^f, then problem $(P^f)_\xi$ possesses a periodic minimizer.*

Remark: If ξ is an extremal point, then either it is an exposed point, or it is an end point of the (non-degenerate) interval of linearity $\mathcal{E}_f^*(\xi)$. In the first case, the existence of a periodic minmizer of $(P^f)_\xi$ was established in [4, Lemma 3.1]. In the second case, this result was stated in [4, Lemma 3.2] but the proof was not complete. Specifically, the proof relied on a statement (based on an argument of [7]) which may not be valid without some additional assumptions. The proof was completed in [9] for integrands of the form (0.5). Here we establish the result in the general case.

The proof of the theorem is based on the following lemma.

LEMMA 2.2: *Suppose that $f \in \mathfrak{M}$ and that ξ_0 is a point such that*

$$\varphi^f(\xi_0) < f(\xi_0, 0, 0). \tag{2.6}$$

Further suppose that there exists a sequence $\{\xi_n\}$ *converging to* ξ_0 *such that* ξ_n *is an exposed point relative to* φ^f *and* $\xi_n \neq \xi_0$, *for each* n. *Then there exists* N *such that*

$$\sup\{\tau(w)\colon w \in \tilde{\mathcal{S}}(\xi_n; f),\ n \geq N\} < \infty. \tag{2.7}$$

Proof: Suppose that the result is not valid. By considering a subsequence if necessary, we may assume that $\{\xi_n\}$ is strictly monotone and that for every n there exists $w_n \in \tilde{\mathcal{S}}(\xi_n; f)$ such that $\tau(w_n) \to \infty$. We shall assume that $\{\xi_n\}$ is monotone decreasing. The proof is similar in the case that it is monotone increasing.

By (2.1) there exists $\lambda_n \in \partial\varphi^f(\xi_n)$ such that $w_n \in \tilde{\mathcal{S}}(f_{\lambda_n})$. To simplify the notation we shall write $f_n := f_{\lambda_n}$. Since φ^f is convex, the sequence $\{\lambda_n\}$ is monotone decreasing and Proposition 0.1 implies that it is bounded. Clearly its limit λ_0 belongs to $\partial\varphi^f(\xi_0)$. In fact λ_0 is the one-sided (right) derivative of φ^f at ξ_0.

We propose to show that

$$\forall \rho > 0,\ \exists \eta \in [\xi_0 - \rho, \xi_0]\colon \varphi^f(\eta) = f(\eta, 0, 0). \tag{2.8}$$

Obviously this contradicts (2.6).

By assumption $f \in \mathfrak{M}(\alpha, \beta, \gamma, \bar{a})$. Clearly there exists $\bar{b} \in \mathbb{R}^4$ such that $f_n \in \mathfrak{M}(\alpha, \beta, \gamma, \bar{b})$, $n = 0, 1, 2, \ldots$ and $f_n \to f_0 := f^{\lambda_0}$ in this space. Consequently, the argument used in the proof of [10, Prop. 2.3] shows that

$$\sup_n \|w_n\|_{W^{1,\infty}(\mathbb{R})} \leq M < \infty. \tag{2.9}$$

Since $w_n \in \tilde{\mathcal{S}}(f_n)$, it follows that w_n is **c-optimal** relative to f_n. This means that, for every bounded interval $D = (T_1, T_2) \subset \mathbb{R}_+$,

$$J^{f_n}(D, w_n) = \hat{\mu}^{f_n}_{|D|}(X_{w_n}(T_1),\ X_{w_n}(T_2)). \tag{2.10}$$

Clearly, (0.9)(iv) holds uniformly in $\mathcal{F} = \{f_n\}$. Therefore, by Theorem 1.1(i), (2.9) and (2.10) it follows that, for every $T > 0$, there exists a constant $c(T)$ depending continuously on T such that

$$\|w_n\|_{W^{2,\gamma}(D)} \leq c(T), \quad \forall D = (s, s+T) \subset \mathbb{R}_+,\ n = 1, 2, \ldots. \tag{2.11}$$

If w_n is a constant, then its value is ξ_n and $\varphi^f(\xi_n) = f(\xi_n, 0, 0)$. Therefore (2.6) implies that for sufficiently large n, w_n is not a constant. Without loss of generality we assume that $w_n(0) = \inf_{\mathbb{R}} w_n < \xi_n$ for all n. Put $\tau_n = \tau(w_n)$ (=the minimal period of w_n). By [10, Lemma 3.1] there exists $\bar{\tau}_n \in (0, \tau(w_n))$ such that w_n is strictly increasing in $(0, \bar{\tau}_n)$ and strictly decreasing in $(\bar{\tau}_n, \tau(w_n))$. Consider the interval

$$K_n = \{t \in [0, \tau_n]\colon w_n(t) \geq \xi_n - \rho\},$$

and put $c_n = |K_n|/\tau_n$. Note that

$$\xi_n = \langle w_n \rangle_{(0,\tau_n)} \leq M c_n + (\xi_n - \rho)(1 - c_n),$$

which implies that $\liminf c_n > 0$. Therefore, since $\tau(w_n) \to \infty$, it follows that $|K_n| \to \infty$. Let $K_n = [\kappa_n, \kappa_n^*]$. Then either $\{\bar{\tau}_n - \kappa_n\}$ or $\{\kappa_n^* - \bar{\tau}_n\}$ or both are unbounded. In the remaining part of the proof we assume that $\{\bar{\tau}_n - \kappa_n\}$ is unbounded. The same argument works if $\{\kappa_n^* - \bar{\tau}_n\}$ is unbounded.

By considering a subsequence if necessary, we may assume that $h_n = \bar{\tau}_n - \kappa_n \to \infty$. Put

$$v_n(t) = w_n(t + \kappa_n), \quad \forall t \geq 0.$$

The boundedness of $\{w_n\}$ (see (2.11)) and standard compactness results imply that there exists a subsequence of $\{v_n\}$ which converges in $C^1(D)$ and weakly in $W^{2,\gamma}(D)$, for every bounded interval $D \subset \mathbb{R}_+$. We shall assume that the full sequence $\{v_n\}$ converges in this sense and denote its limit by v. By the weak lower semi-continuity of the functional $I^{f_0}(D; \cdot)$ (see [2]) it follows that

$$I^{f_0}(D; v) \leq \liminf_{n\to\infty} I^{f_0}(D; v_n) = \liminf_{n\to\infty} I^{f_n}(D; v_n), \tag{2.12}$$

for every bounded interval $D \subset \mathbb{R}_+$. Put $x = X_v(T_1)$, $y = X_v(T_2)$ where T_1, T_2 are the end points of D. The equi-continuity of the sequence of response functions $\{\hat{\mu}_T^{f_n}\}_{n=1}^{\infty}$ together with (2.10) imply that

$$\lim_{n\to\infty} \left(J^{f_n}(D, v_n) - \hat{\mu}_{|D|}^{f_n}(x, y)\right) = 0.$$

In addition,

$$\lim_{n\to\infty} \hat{\mu}_{|D|}^{f_n}(x, y) = \hat{\mu}_{|D|}^{f_0}(x, y).$$

This is an immediate consequence of the uniform boundedness of minimizers of the sequence of problems $\left(P_D^{f_n}\right)^{x,y}, n = 1, 2, \ldots$ (see Theorem 1.1(i)). Hence

$$\lim_{n\to\infty} J^{f_n}(D, v_n) = \hat{\mu}_{|D|}^{f_0}(x, y),$$

and consequently, by (2.12),

$$J^{f_0}(D;v)=\lim_{n\to\infty}J^{f_n}(D;v_n)=\hat{\mu}^{f_0}_{|D|}(x,y). \tag{2.13}$$

Thus v is c-optimal relative to f_0. Since v_n, $n=1,2,\dots$, is monotone increasing in $(0,h_n)$ and $h_n\to\infty$, it follows that v is non-decreasing in $\mathbb{R}_+$. As v is bounded, it follows that it has a limit at $+\infty$,

$$\eta:=\lim_{t\to\infty}v(t)\ge\xi_0-\rho. \tag{2.14}$$

Furthermore, we claim that

$$\lim_{t\to 0}v'(t)=0. \tag{2.15}$$

To verify this claim, let $T>0$ and put $z_n(t)=v(n+t)$, $t\in(0,T)$. Since $\{z_n\}$ is bounded in $C^1[0,T]$ we conclude (by the same argument that was used in proving (2.11)) that $\{z_n\}$ is bounded in $W^{2,\gamma}(0,T)$. Therefore a subsequence $\{z_{n_k}\}$ converges weakly in this space to a function z. Clearly this function has the constant value η. Therefore $z_n\to z$ weakly in $W^{2,\gamma}(0,T)$ and hence in $C^1[0,T]$. This implies (2.15).

Since v is a minimizer of $\left(P_+^{f_0}\right)$ and $\langle v\rangle=\eta$, it follows that it is also a minimizer of $\left(P_+^f\right)_\eta$ and

$$\mu^{f_0}=\varphi^f(\eta)-\lambda_0\eta. \tag{2.16}$$

We claim that

$$\varphi^f(\eta)=J_+^f(v)=f(\eta,0,0). \tag{2.17}$$

By [10, Prop. 2.3], since v is c-optimal relative to f_0, there exists a constant c_0 such that

$$\left|J^{f_0}(D;v)-\mu^{f_0}\right|\le\frac{c_0}{|D|}, \tag{2.18}$$

for every bounded interval D. Therefore, by [10, Lemma 2.5], there exists an f_0-perfect minimizer u such that

$$\{(u,u')(t):t\in\mathbb{R}_+\}\subseteq\Omega(v)=\{(\eta,0)\}.$$

(Here, as in [10], $\Omega(v)$ denotes the limit set of (v,v').) Thus u is the constant function with value η. Since it is a minimizer of $\left(P_+^{f_0}\right)$, we conclude that

$$\mu^{f_0}=f_0(\eta,0,0)=f(\eta,0,0)-\lambda_0\eta.$$

This fact and (2.16) imply (2.17).

By (2.2), $\lambda_0 \in \partial\varphi^f(\eta)$. Since $\{\lambda_n\}$ is strictly decreasing to λ_0, which is the one-sided (right) derivative of φ^f at ξ_0, it follows that $\eta \le \xi_0$. This fact together with (2.14) and (2.17) yield (2.8) and the proof is complete. ∎

Proof of Theorem 2.1: If ξ is an exposed point, the existence of a periodic minimizer of $(P^f)_\xi$ is known (see [4]). Therefore we assume that ξ is not an exposed point. Since it is an extremal point, it follows that ξ is an end point of the (non-degenerate) interval of linearity $\mathcal{E}_f^*(\xi)$ and that φ^f is differentiable at ξ. As in the proof of [4, Lemma 3.2], it can be shown that in this case there exists a sequence $\{\xi_n\}$ with the properties mentioned in Lemma 2.2. If $\varphi^f(\xi) = f(\xi, 0, 0)$, then of course the constant function with value ξ is a periodic minimizer. Therefore we may assume that (2.6) holds. Let $w_n \in \tilde{\mathcal{S}}(\xi_n; f)$ and λ_n, $n = 1, 2, \dots$ be as in the proof of Lemma 2.2 with $\lambda_n \to \lambda_0 = (\varphi^f)'(\xi)$. The lemma implies that the sequence $\{\tau(w_n)\}$ is bounded and we may assume that it converges, say to τ_0. As shown in the proof of the lemma, the sequence $\{w_n\}$ is uniformly bounded in $C^1(\mathbb{R})$ and in $W^{2,\gamma}_{loc}(\mathbb{R})$. Therefore there exists a subsequence which converges uniformly on every compact subset, and weakly in $W^{2,\gamma}_{loc}(\mathbb{R})$ to a periodic function w with period τ_0. It follows that w is a minimizer of problem $\left(P^{f_{\lambda_0}}\right)$ and that

$$\langle w \rangle = \frac{1}{\tau_0} \int_0^{\tau_0} w = \xi.$$

Thus w is a minimizer of problem $\left(P^f\right)_\xi$. ∎

3. Uniform boundedness of approximate minimizers

In this section we prove Theorem I. Its proof is based on several lemmas.

LEMMA 3.1: *Assume that $f \in \mathfrak{M}$ and δ, τ are positive numbers. Then there exists a number $M = M(f, \tau, \delta)$ such that, for every $T \ge \tau$ and every δ-approximate minimizer v of $\left(P^f_{(0,T)}\right)$,*

$$\|v\|_{C^1[0,T]} \le M \quad \text{and} \quad \sup_{0 \le s \le T-1} I^f((s, s+1); v) \le M. \tag{3.1}$$

Proof: The lemma can be established by essentially the same argument as in the proof of Lemma 2.2 of [8]. For the convenience of the reader we present the proof here.

Without loss of generality we may assume that $T > 1$. Indeed for $T \in [\tau, 1]$, (3.1) follows from the coercivity condition (0.11). Put $T = n + b$, where n is an integer and $0 < b < 1$. Consider the partition of $(0,T)$ into subintervals $\{K_j\}_0^{n_1}$ where $K_j = [j, j+1)$ if $j < n-1$ and $K_{n-1} = [n-1, n+b]$.

Let v be a δ-approximate minimizer:

$$I^f((0,T);v) \leq T\mu_T^f + \delta. \tag{3.2}$$

Put $\tau(j) = j$ for $j = 0, \dots, n-1$ and $\tau(n) = T$. Choose a positive number R_0 such that, for any interval D with $1 \leq |D| \leq 2$,

$$u \in W^{2,\gamma}(D), \quad \sup_D |X_u| \geq R_0 \Longrightarrow I^f(D;u) \geq I^f(D,0) + 1. \tag{3.3}$$

The existence of such a number R_0 is guaranteed by the coercivity condition (0.11). Put

$$m_0 = \sup\{|\hat{\mu}_T^f(x,y)|: 1 < T < 2, \ |x| \leq R_0, \ |y| \leq R_0\}. \tag{3.4}$$

Let R_1 be a positive number such that, for any interval D with $1 \leq |D| \leq 2$,

$$u \in W^{2,\gamma}(D), \quad \sup_D |X_u| \geq R_1 \Longrightarrow I^f(D;u) \geq 2(m_0 + b_0) + \delta + 1, \tag{3.5}$$

where b_0 is the constant involved in the coercivity condition (0.11).

Suppose that there exists j such that

$$\sup_{K_j} |X_v| \geq R_1. \tag{3.6}$$

We shall show that this leads to a contradiction. Let j_1, j_2 be two integers such that $0 \leq j_1 \leq j \leq j_2 \leq n$, $\sup_{K_i} |X_v| \geq R_0$ for $j_1 \leq i \leq j_2$, and $D = [\tau(j_1), \tau(j_2)]$ is the maximal interval satisfying these conditions. Put $\hat{D} = (\tau'(j_1), \tau''(j_2))$, where $\tau'(i) = \max(i-1, 0)$ and $\tau''(i) = \tau(i+1)$, $i < n$ and $\tau''(n) = T$.

Next we associate with each point $\tau(j)$, $j = 0, \dots, n$ a point $\zeta_j \in \mathbb{R}^2$ as follows:

(i) $\zeta_j = (0,0)$, if $\tau(j) \in D$,

(ii) $\zeta_j = X_v(\tau(j))$, if $\tau(j) \notin D$.

Let $\tilde{v} \in W^{2,\gamma}(0,T)$ be a function such that

$$\tilde{v}(t) = v(t), \quad \forall t \in (0,T) \setminus \hat{D},$$

and $\tilde{v}$ is a minimizer of $\left(P_{K_i}^f\right)^{(\zeta_i, \zeta_{i+1})}$ in every interval $K_i \subset \hat{D}$.

By (3.3)–(3.5),

$$I^f(D;\tilde{v}) \leq I^f(D;v) - m_1 - 2(m_0 + b_0) - \delta,$$

where m_1 is the largest integer not exceeding $|D|$, and

$$I^f(\hat{D};\tilde{v}) \leq I^f(\hat{D};v) - m_1 - \delta.$$

Hence, by (3.2),

$$I^f((0,T);\tilde{v}) \leq I^f((0,T);v) - 1 - \delta \leq T\mu_T^f - 1.$$

Obviously this is impossible. This proves the first inequality in (3.1).

To verify the second inequality observe that, for every $s \in [0, T-1]$,

$$(3.7) \qquad I^f((s,s+1);v) \leq \delta + \hat{\mu}_1^f(x,y), \quad \text{where } x = X_v(s),\ y = X_v(s+1).$$

Indeed if (3.7) fails for some $s \in [0, T-1]$, then consider the function $\hat{u}$ given by

$$\hat{u} = u, \ \text{ in } \ [0,s] \cup [s+1,T]; \quad \hat{u} = w, \ \text{ in } \ [s,s+1],$$

where w is a minimizer of $\left(P^f_{(s,s+1)}\right)^{x,y}$. Then $I^f((0,T);\hat{u}) < \varphi_T^f$, which is impossible. Now, the first inequality in (3.1) implies that

$$\sup_{0 \leq s \leq s+1} \hat{\mu}_1^f(X_v(s), X_v(s+1)) \leq M',$$

where M' is a constant which depends only on M. Therefore the second inequality in (3.1) (with an appropriately modified constant M) follows from (3.7). ∎

LEMMA 3.2: *Assume that $f \in \mathfrak{M}$ and let $\tau > 0$. Then there exists a positive constant $C_1 = C_1(f;\tau)$ such that*

$$(3.8) \qquad |\mu_T^f - \mu^f| \leq C_1/T, \quad \forall T > \tau.$$

Proof: Let v^* be a periodic minimizer of $\left(P^f\right)$. Let τ^* be the smallest period of v^* such that $\tau^* \geq \tau$. Let $T = n\tau^* + b$, where n is an integer and $\tau^* < b < b_1 := 2\tau^*$. Then

$$(3.9) \qquad T\mu_T^f \leq I^f((0,T);v^*) \leq T\mu^f + c_1,$$

where

$$c_1 = I^{|f|}((0,b_1);v^*) - b\mu^f.$$

Let v be a minimizer of $\left(P^f_{(0,T)}\right)$. By Lemma 3.1 there exists a constant $M = M(f;\tau)$ (independent of T and v) such that $\|v\|_{C^1[0,T]} \leq M$, for every $T \geq \tau$. Let w_1 (resp. w_2) be a minimizer of $\left(P^f_{(0,\tau)}\right)^{0,x}$ (resp. $\left(P^f_{(0,\tau)}\right)^{y,0}$), where $x = (v,v')(0)$

and $y = (v, v')(T)$, respectively. Then $I^f((0,\tau); w_i)$, $i = 1,2$ is bounded by a constant C' which depends only on M and τ. Hence C' depends only on f and τ. Further, let $\tilde{v}_T$ be a periodic function with period $T + 2\tau$ defined as follows:

$$\tilde{v}_T(t) = \begin{cases} w_1(t), & t \in [0,\tau), \\ v_T(t-\tau), & t \in [\tau, T+\tau], \\ w_2(t-T-\tau), & t \in (T+\tau, T+2\tau]. \end{cases} \tag{3.10}$$

Then

$$\mu^f \leq \frac{1}{T+2\tau} I^f((0,T+2\tau); \tilde{v}_T) \leq \frac{T\mu_T^f + 2C'}{T+2\tau}. \tag{3.11}$$

Finally, inequalities (3.9) and (3.11) imply (3.8). ∎

COROLLARY 3.3: *Assume that $f \in \mathfrak{M}$ and δ, τ are positive numbers. Then there exists $M > 0$ such that, for every $T \geq \tau$ and $v \in W^{2,\gamma}(0,T)$,*

(3.12)

$$I^f((0,T); v) \leq T\mu^f + \delta \Longrightarrow \|v\|_{C^1[0,T]} \leq M \text{ and } \sup_{0 \leq s \leq T-1} I^f((s,s+1); v) \leq M.$$

Proof: This is an immediate consequence of the previous two lemmas. ∎

LEMMA 3.4: *Assume that $f \in \mathfrak{M}$ and let τ be a positive number. Then, for every real λ, there exists a constant $C(\lambda) = C(f,\tau;\lambda)$ such that*

$$\xi \in \mathbb{R},\ \lambda \in \partial\varphi^f(\xi) \Longrightarrow |\varphi_T^f(\xi) - \varphi^f(\xi)| \leq C(\lambda)/T, \quad \forall T > \tau. \tag{3.13}$$

Proof: Let f_λ be the function given by $f_\lambda(u,p,r) = f(u,p,r) - \lambda u$. Since $f \in \mathfrak{M}(\alpha,\beta,\gamma,\bar{a})$, it is clear that $f_\lambda \in \mathfrak{M}(\alpha,\beta,\gamma,\bar{a}')$ for some appropriate $\bar{a}'$. Observe that, for every real ξ,

$$\lambda \in \partial\varphi^f(\xi) \Longleftrightarrow \mu^{f_\lambda} = \varphi^{f_\lambda}(\xi) = \varphi^f(\xi) - \lambda\xi.$$

Therefore the statement of the lemma is equivalent to the following:

$$\xi \in \mathbb{R},\ \mu^{f_\lambda} = \varphi^{f_\lambda}(\xi) \Longrightarrow |\varphi_T^{f_\lambda}(\xi) - \varphi^{f_\lambda}(\xi)| \leq C(\lambda)/T, \quad \forall T > \tau. \tag{3.14}$$

To simplify the notation we shall prove this statement for f rather than f_λ.

Put $E = \{\xi : \varphi^f(\xi) = \mu^f\}$. By Proposition 0.1, E is a closed bounded interval which may reduce to a single point. If $\xi \in E$ and there exists a periodic minimizer of $(P^f)_\xi$, then inequality (3.14) follows by the same reasoning as in the first part of the proof of Lemma 3.2. This is always the case if E consists of a single point.

Note that for any $\xi \in \mathbb{R}$ we have $\mu_T^f \le \varphi_T^f(\xi)$. Hence, by Lemma 3.2,

$$\mu^f - \frac{C_1}{T} \le \mu_T^f \le \varphi_T^f(\xi). \tag{3.15}$$

Therefore it remains to prove that there exists a constant C such that, for $\xi \in E$,

$$\varphi_T^f(\xi) \le \mu^f + \frac{C}{T}, \quad \forall T \ge \tau. \tag{3.16}$$

Assume that $E = [\xi_1, \xi_2]$ is a non-degenerate interval. By Theorem 2.1, problem $(P^f)_{\xi_i}$ possesses a periodic minimizer u_i for $i = 1, 2$. Therefore it remains to prove (3.16) for $\xi \in (\xi_1, \xi_2)$. For this purpose we shall construct a function $v \in W^{2,\infty} \cap C^1[0, T]$ such that

$$\langle v \rangle_{(0,T)} = \xi, \quad I^f((0,T); v) \le T\mu^f + C', \tag{3.17}$$

where C' is a constant independent of ξ and T.

Let τ_i be the minimal period of u_i subject to the condition $\tau_i \ge \tau$. We may assume that $T \ge 2(\tau_1 + \tau_2)$. Let $\rho = (\xi_2 - \xi)/(\xi_2 - \xi_1)$ so that $\rho\xi_1 + (1-\rho)\xi_2 = \xi$. Finally let n_1, n_2 be non-negative integers such that $0 < b_j := T_j - n_j\tau_j \le 2\tau_j$ $(j = 1, 2)$ and $\tau \le b = b_1 + b_2$. For $t \in [0, T]$, put

$$v(t) = \begin{cases} u_1(t), & 0 \le t \le n_1\tau_1, \\ w(t - n_1\tau_1), & n_1\tau_1 \le t \le b + n_1\tau_1, \\ u_2(t - b - n_1\tau_1), & b + n_1\tau_1 \le t \le T, \end{cases} \tag{3.18}$$

where w is a minimizer of $(P^f_{(0,b)})^{x,y}_{\xi'}$ with $x = X_{u_1}(0)$, $y = X_{u_2}(0)$ and

$$\xi' = \frac{b_1\xi_1 + b_2\xi_2}{b}.$$

Then, $\langle v \rangle_{(0,T)} = \xi$ and

$$I^f((0,T); v) \le (T - b)\mu^f + I^f((0,b); w) = T\mu^f + C',$$
$$C' = 2(\tau_1 + \tau_2)|\mu^f| + \sup\{\hat{\mu}_S(\eta, x, y) : |\eta| \le |\xi|,\ \tau < S \le 2(\tau_1 + \tau_2)\}.$$

Clearly C' is independent of T so that v satisfies (3.17). This proves (3.16). ∎

Proof of Theorem I: Let v be as in the statement of the theorem. Lemma 3.4 and (0.14) imply that

$$J^f(0, T; v) \le \varphi^f(\xi) + (C(\lambda) + \delta)/T, \quad \forall T \ge \tau. \tag{3.19}$$

Since $\langle v \rangle_{(0,T)} = \xi$ and $\mu^{f_\lambda} = \varphi^f(\xi) - \lambda\xi$, it follows that

$$J^{f_\lambda}(0, T; v) \le \varphi^{f_\lambda}(\xi) + (C(\lambda) + \delta)/T = \mu^{f_\lambda} + (C(\lambda) + \delta)/T, \quad \forall T \ge \tau. \tag{3.20}$$

Hence, by Corollary 3.3, there exists a constant M depending on f, λ, τ, δ, but independent of T, such that (0.15) holds. ∎

4. Uniform distribution of energy and mass

In this section we prove Theorems II and III and discuss their extension to some related problems. We start with

Proof of Theorem III: Let v be as in the statement of the theorem. Then v satisfies (3.20). Theorem I and (1.4) imply that there exists a positive constant C' depending on f_λ such that

$$J^{f_\lambda}(D;v) \geq \mu^{f_\lambda} - \frac{C'}{|D|}, \tag{4.1}$$

for every interval $D \subset (0,T)$.

Now suppose that $T \geq 4\tau$ and that $D = (t_1, t_2)$ is an interval contained in $(0,T)$. Put $D_1 = (0, t_1]$ and $D_2 = [t_2, T)$. Then, by (3.20) and (4.1),

$$I^{f_\lambda}(D;v) + (|D_1| + |D_2|)\mu^{f_\lambda} - 2C' \leq I^{f_\lambda}((0,T);v) \leq T\mu^{f_\lambda} + C(\lambda) + \delta.$$

Consequently

$$I^{f_\lambda}(D;v) \leq |D|\mu^{f_\lambda} + C'', \tag{4.2}$$

where C'' is a constant depending only on f, δ, λ, τ. Combining (4.1) and (4.2) we obtain (0.18). (Recall that under the assumptions of the theorem $\mu^{f_\lambda} = \varphi^f(\xi) - \lambda\xi$.) The second inequality, (0.19), is a special case of (0.18). ∎

For the proof of Theorem II we need an additional lemma.

LEMMA 4.1: *Assume that $f \in \mathfrak{M}$ and let $\tau, \sigma > 0$. Put $B_\sigma = \{x \in \mathbb{R}^2 : |x| < \sigma\}$. Then there exists a positive constant $A = A(f; \tau, \sigma)$ such that*

$$\text{(4.3)}\quad \begin{aligned} &\text{(i)} \quad |\hat{\mu}^f_T(x,y) - \mu^f_T| \leq A/T, \quad \forall T \geq \tau,\ x,y \in B_\sigma, \\ &\text{(ii)} \quad |\hat{\varphi}^f_T(\xi,x,y) - \varphi^f_T(\xi)| \leq A/T, \quad \forall T \geq \tau,\ |\xi| \leq \sigma,\ x,y \in B_\sigma. \end{aligned}$$

Proof: Clearly $\mu^f_T \leq \hat{\mu}^f_T(x,y)$ and $\varphi^f_T(\xi) \leq \hat{\varphi}^f_T(\xi,x,y)$. Therefore, in order to prove (i) and (ii), we only have to show that

$$\hat{\mu}^f_T(x,y) \leq \mu^f_T + A/T, \quad \forall T \geq \tau,\ x,y \in B_\sigma, \tag{4.4}$$

and

$$\hat{\varphi}^f_T(\xi,x,y) \leq \mu^f_T + A/T, \quad \forall T \geq \tau,\ |\xi| \leq \sigma\ x,y \in B_\sigma. \tag{4.5}$$

The functions $T \mapsto \mu_T^f$, $(x,y,T) \mapsto \varphi_T^f(x,y)$ and $(\xi,x,y,T) \mapsto \hat{\varphi}_T^f(\xi,x,y)$ are continuous in $\mathbb{R}_+$, $\mathbb{R}^4 \times \mathbb{R}_+$ and $\mathbb{R}^5 \times \mathbb{R}_+$, respectively (see Theorem 1.1). Therefore, without loss of generality, we shall assume that $\tau \geq 4$.

Let v be a minimizer of $\left(P_{(0,T)}^f\right)$ in case (i), respectively $\left(P_{(0,T)}^f\right)_\xi$ in case (ii), with $T \geq \tau$. By Lemma 3.1 and Theorem I there exists a constant M depending only on f, τ, ξ such that $\|v\|_{C^1[0,T]} \leq M$. Let $U_{\eta,z,\zeta}^T$ be defined as in (A.10) and put

$$\bar{v} := v + U_{0,z,\zeta}^T \quad \text{where } z = x - X_v(0), \;\; \zeta = y - X_v(T).$$

Then $X_{\bar{v}}(0) = x$, $X_{\bar{v}}(T) = y$, $\langle \bar{v} \rangle_{(0,T)} = \langle v \rangle_{(0,T)}$ and consequently

$$\hat{\mu}_T^f(x,y) \leq J^f((0,T);\bar{v}) \leq \mu_T^f + M'/T, \quad \forall x,y \in K,\ T \geq \tau, \tag{4.6}$$

and

$$\hat{\varphi}_T^f(\xi,x,y) \leq J^f((0,T);\bar{v}) \leq \varphi_T^f(\xi) + M'/T, \quad \forall \xi \in [-\sigma,\sigma],\ x,y \in K,\ T \geq \tau, \tag{4.7}$$

where

$$M' = \int_{[0,1]\cup[T-1,T]} \left(\left|f(\bar{v},\bar{v}',\bar{v}'') - f(v,v',v'')\right)\right| dt.$$

In view of (0.9) it is easily seen that M' depends only on f, M, K and therefore only on f, τ, ξ, K. ∎

COROLLARY 4.2: *Under the assumptions of the lemma, there exists a positive constant $b_1 = b_1(f;\tau,\sigma)$ such that*

$$|\hat{\mu}_T^f(x,y) - \mu^f| \leq b_1/T, \quad \forall T \geq \tau,\ x,y \in B_\sigma. \tag{4.8}$$

In addition, for every real ξ, there exists a constant $b_2 = b_2(f;\tau,\sigma,\xi)$ such that

$$|\hat{\varphi}_T^f(\xi,x,y) - \varphi^f(\xi)| \leq b_2/T, \quad \forall T \geq \tau,\ x,y \in B_\sigma. \tag{4.9}$$

Proof: This is an immediate consequence of Lemmas 3.2, 3.4 and 4.1. ∎

Proof of Theorem II: Let $\mathcal{E}_f(\lambda) = [\xi_1,\xi_2]$. (The interval may, of course, reduce to a single point.) Suppose that the first inequality in (0.16) fails. Then there exists a positive number ϵ, a sequence $\{T_n\}$ tending to infinity and a sequence $\{v_n\}$ such that, for each n, v_n is a δ-approximate minimizer of problem $\left(P_{(0,T_n)}^f\right)_\xi$ and there exists an interval $D_n \subset (0,T_n)$ satisfying

$$|D_n| \to \infty \;\text{ and }\; b_n = \langle v_n \rangle_{D_n} \to b \notin [\xi_1 - \epsilon, \xi_2 + \epsilon]. \tag{4.10}$$

By Theorem I, there exists a constant $C = C(\lambda; \delta, \tau; f)$ such that v_n satisfies (0.15).

For each n, v_n is a δ-approximate minimizer of problem $\left(P^f_{D_n}\right)^{x_n, y_n}_{b_n}$ where $D_n = (T_{n,1}, T_{n,2})$ and $x_n = X_{v_n}(T_{n,1})$, $y_n = X_{v_n}(T_{n,2})$. This is proved by the same argument as in the proof of Lemma 3.1. Since $|x_n|, |y_n|, |b_n| \le C$, Lemma 4.1 implies that there exists a positive number δ_1 independent of n (depending only on f, τ, δ and C) such that v_n is a δ_1-approximate minimizer of problem $\left(P^f_{D_n}\right)_{b_n}$, i.e.

$$J^f(D_n; v_n) \le \varphi^f_{\tau_n}(b_n) + \delta_1/\tau_n, \quad \text{where } \tau_n = |D_n|. \tag{4.11}$$

Let u_n be a minimizer of problem $\left(P^f_{D_n}\right)_b$. By Theorem I the sequence $\{u_n\}$ is uniformly bounded as in (0.15). As mentioned before, $\{v_n\}$ is also uniformly bounded in this sense. Therefore we can apply the argument used in the first part of the proof of Theorem 1.1 to derive the following:

$$\left|\varphi^f_{\tau_n}(b_n) - \varphi^f_{\tau_n}(b)\right| \le c_1 |b - b_n|, \tag{4.12}$$

where c_1 depends on the uniform bounds for $\{v_n\}$ and $\{u_n\}$, on b, δ_1 and f but not on n. Hence, by Lemma 3.4,

$$\lim \varphi^f_{\tau_n}(b_n) = \varphi^f(b). \tag{4.13}$$

Since $\varphi^f_{\tau_n}(b_n) \le J^f(D_n; v_n)$, (4.11) and (4.12) imply that

$$\lim J^f(D_n; v_n) = \varphi^f(b). \tag{4.14}$$

Therefore,

$$J^{f_\lambda}(D_n; v_n) \to \varphi^{f_\lambda}(b) \tag{4.15}$$

with f_λ as in the proof of Lemma 3.4. On the other hand, by Theorem III, specifically by (0.18),

$$J^{f_\lambda}(D_n; v_n) \to \varphi^{f_\lambda}(\xi). \tag{4.16}$$

Thus $\varphi^{f_\lambda}(b) = \varphi^{f_\lambda}(\xi)$, which implies that $b \in \mathcal{E}_f(\lambda)$, in contradiction to (4.10). This contradiction proves the first inequality in (0.16). In view of Theorem III, the second inequality follows from the first. ∎

As a consequence of Lemma 4.1 we obtain the following extension of Theorems I–III.

THEOREM 4.3: *Let $f \in \mathfrak{M}$. Given $\lambda \in \mathbb{R}$ and positive numbers δ, τ, σ there exists a number $C = C_f(\lambda;\ \delta, \tau, \sigma)$ such that the following statement holds.*

Let $v \in W^{2,\gamma}(0,T)$, $\xi = \langle v \rangle_{(0,T)}$ and $\lambda \in \partial\varphi^f(\xi)$. If $T \geq \tau$ and $x, y \in B_\sigma$, and if v satisfies

$$J^f(0,T;v) \leq \hat{\varphi}^f_T(\xi,x,y) + \delta/T, \tag{4.17}$$

i.e. v is a δ-approximate minimizer of $\left(P^f_{(0,T)}\right)^{x,y}_\xi$, then v satisfies (0.15) and (0.18). In addition, given $\epsilon > 0$, there exists a number $L = L_f(\lambda; \epsilon, \delta, \sigma)$ such that, if $T \geq L$ and v satisfies the above assumptions, then (0.16) holds.

Finally, if v is a δ-approximate minimizer of $\left(P^f_{(0,T)}\right)^{x,y}$, then it will satisfy (4.17) and consequently the above statements will apply to it.

Proof: Lemma 4.1 implies that if v satisfies the conditions of the theorem, then it is a $\bar{\delta}$-approximate minimizer of problem $\left(P^f_{(0,T)}\right)_\xi$, where $\bar{\delta} = \delta + 2A$. ($A$ will also depend on λ, or, more precisely, on a bound for $\mathcal{E}_f(\lambda)$.) Therefore the stated result follows immediately from Theorems I–III. ∎

A. Appendix

In this appendix we provide a proof of Lemma 1.2.

Without loss of generality, we shall assume that $\tau > 4$, where τ is the left end point of D. By rescaling we can always reduce the situation to this case. Put $\mathcal{G}_i = K_i \times D \times \mathcal{F}$.

For $T \geq 1$ we have $b_0 \leq \mu^f_T(\xi) \leq f(\xi, 0, 0)$ with b_0 as in (0.11). Since b_0 is independent of $f \in \mathfrak{M}$ it follows that $\{\Lambda^f_1(\xi,T) : (\xi,T,f) \in K_1 \times D \times \mathcal{F}_0\}$ is bounded. Therefore statement (i) follows from the coercivity inequality (0.11).

Let $\xi_j \in \mathbb{R}$ and let $u_j \in \mathcal{S}^f_T(\xi_j)$ $(j = 1, 2)$. Then the function $\bar{u}_1 = u_1 + (\xi_2 - \xi_1)$ satisfies $\langle \bar{u}_1 \rangle_{(0,T)} = \xi_2$ and consequently

$$J^f((0,T);\bar{u}_1) \geq \Lambda^f_1(\xi_2, T).$$

On the other hand,

$$\left| J^f((0,T);\bar{u}_1) - \Lambda^f_1(\xi_1,T) \right| \leq A_1(f,T)/T,$$

where

$$A_1(f,T) := \int_0^T |f(u_1, u_1', u_1'') - f(u_1 + (\xi_2 - \xi_1), u_1', u_1'')|\, dt. \tag{A.1}$$

Hence, $\Lambda_1^f(\xi_1, T) + A_1/T \geq \Lambda_1^f(\xi_2, T)$. Similarly, we obtain

$$\Lambda_1^f(\xi_2, T) + A_2(f,T)/T \geq \Lambda_1^f(\xi_1, T)$$

where

$$A_2(f,T) = \int_0^T |f(u_2, u_2', u_2'') - f(u_2 - (\xi_2 - \xi_1), u_2', u_2'')| \, dt. \tag{A.2}$$

Consequently,

$$\left|\Lambda_1^f(\xi_1, T) - \Lambda_1^f(\xi_2, T)\right| \leq (A_1 + A_2)/T. \tag{A.3}$$

By (A.1) and (0.9), if $u \in \mathcal{S}_T^f(\xi)$ is a minimizer of $\left(P_{(0,T)}^f\right)_\xi$ and $\eta \in \mathbb{R}$,

$$\int_0^T |f(u, u', u'') - f(u + \eta, u', u'')| \, dt \leq C_0|\eta| \int_0^T (1 + |u''|^\gamma) \, dt \leq C_1|\eta|, \tag{A.4}$$

where C_0, C_1 are constants independent of ξ, T, f in $\mathcal{G}_1$. This inequality and (A.3) imply (1.4) for $i = 1$.

Let $V \in C^2([0,1] \times \mathbb{R}^2)$ be a function such that

$$(V, V')(0, z) = z, \quad (V, V')(1, z) = 0, \quad V(t, 0) \equiv 0, \tag{A.5}$$

where $V' = \partial V/\partial t$. In fact, one can construct a function V possessing these properties, such that $V(\cdot, x)$ is a polynomial of order 3 with coefficients in $C^\infty(\mathbb{R}^2)$ vanishing at zero. For every $x, y \in \mathbb{R}^2$ and $T > 2$, let $V_{x,y}^T$ be a function on $[0, T]$ defined as follows:

$$V_{x,y}^T(t) = \begin{cases} V(t, x), & 0 \leq t \leq 1, \\ 0, & 1 \leq t \leq T - 1, \\ V(T - t, \bar{y}), & T - 1 \leq t \leq T, \end{cases} \tag{A.6}$$

where (for $y = (r_1, r_2)$) $\bar{y} = (r_1, -r_2)$.

For $T > 2$ we have $b_0 \leq \hat{\mu}_T^f(x, y) \leq J^f((0,T); V_{x,y}^T)$, with b_0 as in (0.11). Therefore $\{\Lambda_2^f(x, y, T) : (x, y, T, f) \in K_2 \times D \times \mathcal{F}_0\}$ is bounded. As before, this fact and (0.11) imply statement (i) for $i = 2$.

Given two pairs of points in $\mathbb{R}^2$, say x_1, y_1 and x_2, y_2, let u_j be a minimizer of $(P_{(0,T)}^f)^{x_j, y_j}$, $j = 1, 2$. Put

$$\bar{u}_1(t) = u_1 + V_{\hat{x},\hat{y}}^T, \quad \bar{u}_2(t) = u_2 - V_{\hat{x},\hat{y}}^T,$$

where $\hat{x} = x_2 - x_1$, $\hat{y} = y_2 - y_1$. Then

$$X_{\bar{u}_1}(0) = x_2,\ X_{\bar{u}_1}(T) = y_2; \quad X_{\bar{u}_2}(0) = x_1,\ X_{\bar{u}_2}(T) = y_1,$$

and consequently

$$J^f((0,T);\bar{u}_1) \geq \Lambda_2^f(x_2, y_2, T), \quad J^f((0,T);\bar{u}_2) \geq \Lambda_2^f(x_1, y_1, T).$$

On the other hand, for $j = 1, 2$,

$$\left| J^f((0,T);\bar{u}_j) - \Lambda_2^f(x_j, y_j, T) \right| \leq B_j(f,T)/T,$$

where

$$B_j(f,T) = \int_{[0,1]\cup[T-1,T]} \left| f(u_j, u_j', u_j'') - f(\bar{u}_j, \bar{u}_j', \bar{u}_j'') \right| dt.$$

Hence

$$\left| \Lambda_2^f(x_1, y_1, T) - \Lambda_2^f(x_2, y_2, T) \right| \leq (B_1 + B_2)/T. \tag{A.7}$$

Put $\eta = |x_2 - x_1| + |y_2 - y_1|$. As before (see (A.4)) we obtain

$$B_j \leq C_0\eta(\|V(\cdot, z\|_{C^2[0,1]} + \|V(\cdot, \zeta\|_{C^2[0,1]}) \int_{[0,1]\cup[T-1,T]} (1 + |u_j''|^\gamma)\, dt \leq C_1\eta, \tag{A.8}$$

where c_0, C_1 are constants independent of x_j, y_j, T, f in $\mathcal{G}_2$. This implies (1.4) for $i = 2$.

Let $U \in C^2([0,1] \times \mathbb{R}^4)$ be a function such that (for $z, \zeta \in \mathbb{R}^2$)

$$(U, U')(0, z, \zeta) = z, \quad (U, U')(1, z, \zeta) = \zeta, \quad U(t, 0, 0) \equiv 0, \quad \int_0^1 U(t, z, \zeta)\, dt = 0, \tag{A.9}$$

where $U' = \partial U/\partial t$. One can construct a function U satisfying these conditions such that $U = U(t, z, \zeta)$ is a fourth order polynomial with respect to t with coefficients depending smoothly on z, ζ.

For $z, \zeta \in \mathbb{R}^2$ and $\eta \in \mathbb{R}$, let $U^T_{\eta,z,\zeta}$ be the function given by

$$U^T_{\eta,z,\zeta}(t) = \begin{cases} U(t, z, (\eta, 0)), & 0 \leq t \leq 1, \\ (1 + 2/T)\eta, & 1 \leq t \leq T - 1, \\ U(t - T + 1, (\eta, 0), \zeta), & T - 1 \leq t \leq T. \end{cases} \tag{A.10}$$

Observe that $u = U^T_{\eta,z,\zeta} \in W_{2,\infty}(0,t)$ and

$$\langle u \rangle_{(0,T)} = \eta, \quad (u, u')(0) = z, \quad (u, u')(T) = \zeta.$$

Therefore

$$b_0 \leq \mathring{\varphi}_T^f(\xi, z, \zeta) \leq J^f((0,T); U^T_{\xi,z,\zeta}).$$

Consequently, $\{\Lambda_3^f(\xi, z, \zeta, T)\colon (\xi, z, \zeta) \in K_3,\ T \in D,\ f \in \mathcal{F}_0\}$ is bounded. As before, this fact and (0.11) imply statement (i) for $i = 3$.

Given $x_1, y_1, x_2, y_2 \in \mathbb{R}^2$ and $\xi_1, \xi_2 \in \mathbb{R}$ let $u_j \in \mathcal{S}_T^f(\xi_j, x_j, y_j)$, $j = 1, 2$. Put

$$\bar{u}_1(t) = u_1 + U^T_{\eta,z,\zeta},\ \bar{u}_2(t) = u_2 - U^T_{\eta,z,\zeta} \text{ with } \eta = \xi_2 - \xi_1,\ z = x_2 - x_1,\ \zeta = y_2 - y_1.$$

With this notation, the proof of (1.4) for $i = 3$ is completed as in the previous cases. ∎

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